

**SECTION 1 (Maximum Marks: 18)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:  
*Full Marks* : +3 If **ONLY** the correct option is chosen;  
*Zero Marks* : 0 If none of the options is chosen (i.e. the question is unanswered);  
*Negative Marks* : -1 In all other cases.

- Q.1 Suppose  $a, b$  denote the distinct real roots of the quadratic polynomial  $x^2 + 20x - 2020$  and suppose  $c, d$  denote the distinct complex roots of the quadratic polynomial  $x^2 - 20x + 2020$ . Then the value of

$$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$$

is

- (A) 0                      (B) 8000                      (C) 8080                      (D) 16000
- Q.2 If the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = |x|(x - \sin x)$ , then which of the following statements is **TRUE**?
- (A)  $f$  is one-one, but **NOT** onto  
 (B)  $f$  is onto, but **NOT** one-one  
 (C)  $f$  is **BOTH** one-one and onto  
 (D)  $f$  is **NEITHER** one-one **NOR** onto

- Q.3 Let the functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = e^{x-1} - e^{-|x-1|} \quad \text{and} \quad g(x) = \frac{1}{2}(e^{x-1} + e^{1-x}).$$

Then the area of the region in the first quadrant bounded by the curves  $y = f(x)$ ,  $y = g(x)$  and  $x = 0$  is

- (A)  $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$                       (B)  $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$   
 (C)  $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$                       (D)  $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

Q.4 Let  $a, b$  and  $\lambda$  be positive real numbers. Suppose  $P$  is an end point of the latus rectum of the parabola  $y^2 = 4\lambda x$ , and suppose the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  passes through the point  $P$ . If the tangents to the parabola and the ellipse at the point  $P$  are perpendicular to each other, then the eccentricity of the ellipse is

- (A)  $\frac{1}{\sqrt{2}}$                       (B)  $\frac{1}{2}$                       (C)  $\frac{1}{3}$                       (D)  $\frac{2}{5}$

Q.5 Let  $C_1$  and  $C_2$  be two biased coins such that the probabilities of getting head in a single toss are  $\frac{2}{3}$  and  $\frac{1}{3}$ , respectively. Suppose  $\alpha$  is the number of heads that appear when  $C_1$  is tossed twice, independently, and suppose  $\beta$  is the number of heads that appear when  $C_2$  is tossed twice, independently. Then the probability that the roots of the quadratic polynomial  $x^2 - \alpha x + \beta$  are real and equal, is

- (A)  $\frac{40}{81}$                       (B)  $\frac{20}{81}$                       (C)  $\frac{1}{2}$                       (D)  $\frac{1}{4}$

Q.6 Consider all rectangles lying in the region

$$\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2 \sin(2x) \right\}$$

and having one side on the  $x$ -axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

- (A)  $\frac{3\pi}{2}$                       (B)  $\pi$                       (C)  $\frac{\pi}{2\sqrt{3}}$                       (D)  $\frac{\pi\sqrt{3}}{2}$

**SECTION 2 (Maximum Marks: 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:
 

<i>Full Marks</i>	: +4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	: +3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	: 0	If none of the options is chosen (i.e. the question is unanswered);
<i>Negative Marks</i>	: -2	In all other cases.

- Q.7 Let the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^3 - x^2 + (x - 1) \sin x$  and let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function. Let  $fg: \mathbb{R} \rightarrow \mathbb{R}$  be the product function defined by  $(fg)(x) = f(x)g(x)$ . Then which of the following statements is/are TRUE?
- (A) If  $g$  is continuous at  $x = 1$ , then  $fg$  is differentiable at  $x = 1$
- (B) If  $fg$  is differentiable at  $x = 1$ , then  $g$  is continuous at  $x = 1$
- (C) If  $g$  is differentiable at  $x = 1$ , then  $fg$  is differentiable at  $x = 1$
- (D) If  $fg$  is differentiable at  $x = 1$ , then  $g$  is differentiable at  $x = 1$
- Q.8 Let  $M$  be a  $3 \times 3$  invertible matrix with real entries and let  $I$  denote the  $3 \times 3$  identity matrix. If  $M^{-1} = \text{adj}(\text{adj } M)$ , then which of the following statements is/are ALWAYS TRUE?
- (A)  $M = I$                       (B)  $\det M = 1$                       (C)  $M^2 = I$                       (D)  $(\text{adj } M)^2 = I$
- Q.9 Let  $S$  be the set of all complex numbers  $z$  satisfying  $|z^2 + z + 1| = 1$ . Then which of the following statements is/are TRUE?
- (A)  $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$  for all  $z \in S$
- (B)  $|z| \leq 2$  for all  $z \in S$
- (C)  $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$  for all  $z \in S$
- (D) The set  $S$  has exactly four elements
- Q.10 Let  $x, y$  and  $z$  be positive real numbers. Suppose  $x, y$  and  $z$  are the lengths of the sides of a triangle opposite to its angles  $X, Y$  and  $Z$ , respectively. If

$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x + y + z},$$

then which of the following statements is/are TRUE?

- (A)  $2Y = X + Z$                       (B)  $Y = X + Z$
- (C)  $\tan \frac{X}{2} = \frac{x}{y+z}$                       (D)  $x^2 + z^2 - y^2 = xz$

Q.11 Let  $L_1$  and  $L_2$  be the following straight lines.

$$L_1: \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \quad \text{and} \quad L_2: \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}.$$

Suppose the straight line

$$L: \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing  $L_1$  and  $L_2$ , and passes through the point of intersection of  $L_1$  and  $L_2$ . If the line  $L$  bisects the acute angle between the lines  $L_1$  and  $L_2$ , then which of the following statements is/are TRUE?

- (A)  $\alpha - \gamma = 3$       (B)  $l + m = 2$       (C)  $\alpha - \gamma = 1$       (D)  $l + m = 0$

Q.12 Which of the following inequalities is/are TRUE?

(A)  $\int_0^1 x \cos x \, dx \geq \frac{3}{8}$

(B)  $\int_0^1 x \sin x \, dx \geq \frac{3}{10}$

(C)  $\int_0^1 x^2 \cos x \, dx \geq \frac{1}{2}$

(D)  $\int_0^1 x^2 \sin x \, dx \geq \frac{2}{9}$

**SECTION 3 (Maximum Marks: 24)**

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:  
*Full Marks* : +4 If ONLY the correct numerical value is entered;  
*Zero Marks* : 0 In all other cases.

Q.13 Let  $m$  be the minimum possible value of  $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$ , where  $y_1, y_2, y_3$  are real numbers for which  $y_1 + y_2 + y_3 = 9$ . Let  $M$  be the maximum possible value of  $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$ , where  $x_1, x_2, x_3$  are positive real numbers for which  $x_1 + x_2 + x_3 = 9$ . Then the value of  $\log_2(m^3) + \log_3(M^2)$  is \_\_\_\_\_

- Q.14 Let  $a_1, a_2, a_3, \dots$  be a sequence of positive integers in arithmetic progression with common difference 2. Also, let  $b_1, b_2, b_3, \dots$  be a sequence of positive integers in geometric progression with common ratio 2. If  $a_1 = b_1 = c$ , then the number of all possible values of  $c$ , for which the equality

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer  $n$ , is \_\_\_\_\_

- Q.15 Let  $f: [0, 2] \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right).$$

If  $\alpha, \beta \in [0, 2]$  are such that  $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$ , then the value of  $\beta - \alpha$  is \_\_\_\_\_

- Q.16 In a triangle  $PQR$ , let  $\vec{a} = \overrightarrow{QR}$ ,  $\vec{b} = \overrightarrow{RP}$  and  $\vec{c} = \overrightarrow{PQ}$ . If

$$|\vec{a}| = 3, \quad |\vec{b}| = 4 \quad \text{and} \quad \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|},$$

then the value of  $|\vec{a} \times \vec{b}|^2$  is \_\_\_\_\_

- Q.17 For a polynomial  $g(x)$  with real coefficients, let  $m_g$  denote the number of distinct real roots of  $g(x)$ . Suppose  $S$  is the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial  $f$ , let  $f'$  and  $f''$  denote its first and second order derivatives, respectively. Then the minimum possible value of  $(m_{f'} + m_{f''})$ , where  $f \in S$ , is \_\_\_\_\_

- Q.18 Let  $e$  denote the base of the natural logarithm. The value of the real number  $a$  for which the right hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is \_\_\_\_\_

**END OF THE QUESTION PAPER**